DISSERTATIO MATHEMATICA,
SPECIMINA QUÆDAM
GEOMETRIÆ CURVILINEÆ
SISTENS.

QUAM

Conſ. Ampl. Fac. Philoſ. Aboënſ.

Publico examini ſubjiciunt

MAG. *ANDREAS JOHANNES METHER,*

Math. Applic. Doc. nec non R. Acad. Bibl. Aman. Ord.

ET

NICOLAUS ISRAËL BERGHÆLL,

Stip. Reg. Tavaſtenſes.

In Auditorio Maj. die 3 Novembr. A. 1798.

Horis a. m. ſolitis.

Typis FRENCKELLIANIS.

§. I.

Geometriam Curvilineam invento calculo differentiali & Integrali, insignes utique fecisse progressus, quisque facile concedit; ad illud vero fastigium nondum pervenit, ut Problemata omnia in illa occurrentia methodo quadam generali solvi possint. Quoties enim quantitatum differentiæ in natura Curvarum investiganda adhibeantur, generaliter Problemata tractari possunt; si vero solutio pendeat ex integratione quantitatum differentialium, res æque bene non succedit. Quæ autem jam attulimus, inprimis valent de illa Geometriæ Curvilineæ parte, quæ Methodi Tangentium nomine insigniri solet, & in duas abit partes, alteram videlicet Directam, cujus ope, data æquatione Curvæ, Subtangentes, Tangentes & reliqua ex Tangentibus dependentia inveniuntur, atque alteram hujus Inversam, qua ex dato valore Subtangentis, seu alia quacunque Curvæ proprietate, investigatur natura Curvæ. Directa scilicet methodus ejusmodi jam cepit incrementa, ut nihil fere amplius in illa desiderari videatur: Inversa autem generalis exstitit nulla, ejusque loco particulares tantum dari possunt regulæ. *) Hoc autem ipsius

*) Cfr. JAC. BERNOULLI Opp. Tom. I. pag. 622.

us rei naturæ non tribuatur necesse est, nam difficultas ex integratione quantitatum differentialium est derivanda. In genere etenim integratio æquationum differentialium primi ordinis eadem est ac Methodus Inversa Tangentium. Directa enim in eo inest, ut inveniatur valor Subtangentis $\frac{ydx}{dy}$ seu $\frac{dx}{dy}$ data æquatione Curvæ. Si vero jam detur æquatio differentialis primi ordinis semper dabitur $\frac{dx}{dy}$ vel $\frac{ydx}{dy}$ in functionibus Coordinatarum x & y, quæ æquatio integrata, dabit æquationem Curvæ. Arduam autem hanc persæpe esse integrationem atque difficillimam, satis constat; quum vero Problemata quædam huc pertinentia nobis succurrerint, horum solutionem Tuæ B. L. censuræ submittere jam audemus.

§. II.

PROBLEMA. *Si recta AT, inter punctum datum A & Tangentem MT Curvæ cujusvis LM intercepta dicatur v, & Coordinatæ Orthogonales AP = x & PM = y, ex relatione inter v & alterutram Coordinatarum invenire æquationem Curvæ.*

Sumto puncto p infinite vicino ipsi P, ductisque pm & QM parallelis PM & AP respective, erit $\Delta mMQ \sim \Delta MPT$, adeoque $mQ\ (dy) : QM\ (dx) ::$

$y :$

$y : PT = \frac{ydx}{dy}$ Est autem $AT = PT - AP = \frac{ydx}{dy} - x$, unde $v = \frac{ydx - xdy}{dy}$, quæ æquatio sufficit ad Curvam determinandam, si v fuerit functio quædam Coordinatarum.

Exempl. 1. Si quæratur Curva talis, ut sit $v = nx$, erit hoc ipsius v valore in æquatione $v = \frac{ydx - xdy}{dy}$ substituto, $nxdy + xdy = ydx$, unde reductione obtinetur $\frac{dx}{(n+1)x} = \frac{dy}{y}$, ex qua integrando eruitur $Ly = \frac{1}{n+1} Lx$ (denotante L Logarithmum Hyperbolicum) & transeundo a Logarithmis ad quantitates absolutas $y = x^{\frac{1}{n+1}}$, quæ æquatio, nisi fuerit $n = -1$ exhibet indolem Curvæ. Erit autem hæc ipsa Algebraica, denotante n numerum quendam rationalem; si vero fuerit n irrationalis, erit Curva earum ex numero, quæ interscendentium nomine insigniri solent.

Exempl. 2. Si $v = \frac{\overline{a+x}.x}{\frac{1}{2}a+x} - x$, obtinetur facta substitutione $\frac{\overline{a+x}.xdy}{\frac{1}{2}a+x.} = ydx$; quæ æquatio ad integrationem disposita dabit $\frac{\overline{\frac{1}{2}a+x}.dx}{a+x.} = \frac{dy}{y}$, cujus integrale Logarithmum est $Ly = \frac{1}{2} L(ax + x^2)$, & hinc $y^2 = ax + x^2$, æquatio ad Hyperbolam æquilateralem.

Schol. 1. Potest etjam natura Curvæ inveniri ex relatione data inter NT & alterutram Coordinatarum. Quum enim sit $\Delta MQm \backsim \Delta NPM$, erit $dx : dy :: y\ NP = \frac{ydy}{dx}$ & $PT = \frac{ydx}{dy}$ (§. II.) habebitur itaque $NT = \frac{y.\sqrt{dx^2 + dy^2}}{dxdy}$.

Exempl. 1. Posita $NT = ay^2$, erit $aydxdy = dx^2 + dy^2$. Ad hanc integrandam ponatur $dy = \frac{zdy}{b}$, quo ipsius dx valore in æquatione substituto eruitur $\frac{ayzdy^2}{b} = dy^2 \frac{(z^2 + b^2)}{b^2}$ & facta debita reductione obtinetur $y = \frac{z^2 + b^2}{abz}$ & $dy = \frac{2z^2dz - (z^2 + b^2)dz}{abz^2} = \frac{dz}{ab} - \frac{bdz}{az^2}$. Loco autem dy, si substituatur valor jam inventus in æquatione $\frac{zdy}{b} = dx$, habebitur $\frac{zdz}{ab^2} - \frac{dz}{az} = dx$, eritque hujus integrale $\frac{z^2}{2ab^2} - \frac{1}{a} Lz = x \pm C$. Ipsa autem Curva ope æquationum $y = \frac{z^2 + b^2}{abz}$ & $x = \frac{z^2}{2ab^2} - \frac{1}{a} Lz \pm C$ construi potest. Sumta enim

$AP = z$ ceu axi, erectaque PM perpendiculariter in AP & facta $PM = y = \frac{z^2 + b^2}{abz}$ erit LM Curva quae definitur aequatione $y = \frac{z^2 + b^2}{abz}$. Si vero prolongatur PM ut fiat $PS = x = \frac{z^2}{2ab^2} - \frac{1}{a} Lz \pm C$, habebitur Curva, quae aequatione $x = \frac{z^2}{2ab^2} - \frac{1}{a} Lz \pm C$ determinatur. Ducta praeterea Linea AF parallela ipsi PS & SF parallela AP, producatur SF, ut evadat $FH = PM$, erit punctum H in curva quaesita.

Exempl. 2. Si fuerit $NT = ax$ habebitur $axdxdy = y.\overline{dx^2 + dy^2}$. Positis autem $y = ux$ & $dy = pdx$, abit aequatio facta substitutione & reductione in hanc formam: $ap = u.\overline{1 + p^2}$, unde prodit $u = \frac{ap}{1 + p^2}$, adeoque $du = \frac{adp(1 + p^2 - 2p^2)}{(1 + p^2)^2}$. Est vero $dy = pdx = xdu + udx$, ergo $\frac{dx}{x} = \frac{du}{p - u} = \frac{adp(1 + p^2 - 2p^2)}{p.\overline{1+p^2}^2 - ap(1 + p^2)} =$

$$\frac{adp.\overline{1+p^2}}{p.\overline{1+p^2}^2 - ap.\overline{1+p^2}} - \frac{2ap^2dp}{p.\overline{1+p^2}^2 - ap.\overline{1+p^2}}.$$

Integrale vero membri prioris $\frac{adp.\overline{1+p^2}}{p.\overline{1+p^2}^2 - ap.\overline{1+p^2}} = \frac{adp}{p.\overline{1+p^2} - a}$ ut ut habeatur, ponatur $p^2 + 1 = z$, unde $p = \overline{z - 1}^{\frac{1}{2}}$ & $dp = \frac{dz}{2\sqrt{z-1}}$, atque $\frac{adp}{p(1+p^2-a)} = \frac{adz}{2(z-1)(z-a)}$; quae posita aequalis aequationi fictitiae $\frac{a}{2}.\frac{Adz}{z-1} + \frac{Bdz}{z-a}$, dabit $A = \frac{1}{1-a}$ & $B = \frac{-1}{1-a}$; obtinetur itaque $\frac{a}{2}\frac{dz}{(z-1)(z-a)}$ $= \frac{a}{2}.\frac{1}{1-a}\left(\frac{dz}{z-1} - \frac{dz}{z-a}\right)$, ex qua integrando eruitur $\frac{a}{2(1-a)}L\overline{z-1} - L\overline{z-a} = \frac{a}{2(1-a)}(Lp^2 - L\overline{p^2+1-a})$ existente videlicet $z = p^2 + 1$. Posteriori autem membro $-\frac{2ap^2dp}{p.\overline{1+p^2}^2 - ap.\overline{1+p^2}} = \frac{-2apdp}{\overline{1+p^2}^2 - a\overline{1+p^2}}$ addito $\frac{4p.(1+p^2)dp}{\overline{1+p^2}^2 - a(1+p^2)}$, prodit $\int\frac{4p.\overline{1+p^2}dp - 2apdp}{\overline{1+p^2}^2 - a.\overline{1+p^2}} =$ $L\overline{\overline{1+p^2}^2 - a\overline{1+p^2}}$. Hinc vero subducendum $\int\frac{4p.\overline{1+p^2}dp}{\overline{1+p^2}^2 - a.\overline{1+p^2}} = \int\frac{4pdp}{1+p^2-a} = 2L\overline{1+p^2-a}$. Erat autem $\frac{dx}{x} = \frac{adp.(1+p^2-2p^2)}{p.\overline{1+p^2}^2 - ap+p^2}$, unde $Lx =$ $\frac{a}{2(1-a)}(Lp^2 - L\overline{1+p^2-a}) + L\overline{\overline{1+p^2}^2 - a\overline{1+p^2}}$ $- 2L\overline{1+p^2-a} = \frac{a}{2(1-a)}L\frac{p^2}{(1+p^2-a)} + L\frac{\overline{1+p^2}^2 - a.\overline{1+p^2}}{(1+p^2-a)^2}$

&

& transeundo a Logarithmis ad quantitates absolutas

$x = \left(\frac{p^2}{1+p^2-a}\right)^{\frac{a}{2(1-a)}} \left(\frac{\overline{1+p^2}^2 - a.\overline{1+p^2}}{(1+p^2-a)^2}\right)$, quæ quidem æquatio exhibet indolem Curvæ ex mutua Coordinatarum Orthogonalium relatione dependentem, substituto valore ipsius $p = \frac{ax \pm \sqrt{a^2x^2 - 4y^2}}{2y}$.

Schol. 2. Pariter Curva determinari potest ex relatione inter *AN* & alterutram Coordinatarum data. Erit enim $AN = \frac{ydy}{dx} + x = \frac{ydy + xdx}{dx}$; eodem itaque modo quo jam supra §. II. ostendimus, invenitur Curva.

§. III.

PROBLEMA. *Invenire Curvam, in qua Subnormalis est ad Summam Subnormalis & Subtangentis, ut Normalis ad Radium Curvaturæ, datis harum Linearum relationibus.*

Quum sit Subnormalis $= \frac{ydy}{dx}$ & Subtangens $= \frac{ydx}{dy}$, erit Summa $= \frac{y.\overline{dx^2 + dy^2}}{dxdy}$; existente porro dx constante, habebitur Radius Curvaturæ $= \frac{(dx^2+dy^2)^{\frac{3}{2}}}{-ddydx}$;

 est

est vero Normalis $= \frac{y.\sqrt{dx^2 + dy^2}}{dx}$, adeoque secundum hypothesin $\frac{ydy}{dx} : \frac{y.\overline{dx^2 + dy^2}}{dxdy} : : \frac{y.\sqrt{dx^2 + dy^2}}{dx} : \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{- ddydx}$. Sumendo facta mediorum & ultimorum obtinetur æquatio differentio-differentialis $dy^2 + yddy = 0$, cujus integrale primum est $ydy \pm Cdx = 0$, ex qua iterum integrando eruitur $y^2 \pm 2Cx \mp 2C' = 0$, æquatio ad Parabolam.

Coroll. Hinc itaque liquet, in Parabola Conica Subnormalem esse ad summam Subnormalis & Subtangentis, in eadem ratione ac Normalis ad Radium Curvaturæ. Proprietatem vero hanc ex valore Radii Curvedinis directe deducere possumus. Si enim fuerit $y^2 = px$ æquatio Parabolæ, erit Radius Curvaturæ $R = \frac{(4px + p^2)^{\frac{3}{2}}}{2p^2}$; potest autem hic valor sequenti modo exhiberi: $R = \frac{(4x + p.)}{2p} \sqrt{4px + p^2}$, adeoque $2pR = (4x + p) \sqrt{4px + p^2}$, unde eruitur analogia: $p : 4x + p : : \sqrt{4px + p^2} : 2R$; & si dividatur analogia per 2, erit $\frac{p}{2} : 2x + \frac{1}{2}p : : \sqrt{px + \frac{1}{4}p^2} : R$. Est autem in Parabola, cujus Parameter $= p$, Subnormalis $= \frac{1}{2}p$, Summa Subnormalis & Subtangentis $= 2x + \frac{1}{2}p$, & Normalis $= \sqrt{px + \frac{1}{4}p^2}$.

§. IV.

§. IV.

Problema. *Investigare indolem Curvæ, cujus Radius Curvaturæ est ad productum Normalis & Summæ Subtangentis & Subnormalis, ut Subnormalis ad Ordinatam.*

Existente Summa Subnormalis & Subtangentis $= \frac{y.\overline{dx^2 + dy^2}}{dydx}$ & Normali $= \frac{y.\sqrt{dx^2 + dy^2}}{dx}$, erit $\left(\frac{y.\overline{dx^2 + dy^2}}{dxdy}\right).\left(\frac{y.\sqrt{dx^2 + dy^2}}{dx}\right) = \frac{y^2.\overline{dx^2 + dy^2}^{\frac{3}{2}}}{dx^2dy}$. Jam vero ex hypothesi habebitur $\frac{\overline{dx^2 + dy^2}^{\frac{3}{2}}}{-ddydx} : \frac{y^2.\overline{dx^2 + dy^2}^{\frac{3}{2}}}{dx^2dy} :: \frac{ydy}{dx} : y$, unde facta reductione eruitur æquatio differentialis secundi ordinis $dx^2 + y^2ddy = 0$. (A). Ad hanc integrandam ponatur $dy = zdx$, eritque, posito dx constante, $ddy = dzdx$; valore autem ipsius ddy in æquatione (A) substituendo, habebitur, ipsa insuper per dx divisa, $dx + y^2dz = 0$ (B); quum autem fuerit $dy = zdx$, erit $dx = \frac{dy}{z}$, adeoque, si loco dx in æquatione (B) ponatur $\frac{dy}{z}$, obtinebitur, terminis ad integrationem rite dispositis $\frac{dy}{y^2} = -zdz$ & peracta integratione $\frac{1}{y} = \frac{z^2}{2}$ (C). Sumendo autem

$dy = zdx$, habebitur $z = \frac{dy}{dx}$; si itaque resumatur æquatio (C) & loco ipsius z adhibeatur ejus valor jam determinatus, erit $\frac{1}{y} = \frac{dy^2}{2dx^2}$ & hinc $2dx^2 = ydy^2$ cujus integrale, omissa iterum quantitate constante, $\sqrt{2x} = \frac{2}{3}y^{\frac{3}{2}}$ exhibet indolem curvæ, quæ hoc in casu erit Parabola Semi Cubica.

Schol. Si inferatur $\frac{\overline{dx^2 + dy^2}^{\frac{3}{2}}}{-ddydx} : \frac{y^2 . \overline{dx^2 + dy^2}^{\frac{3}{2}}}{dx^2 dy} :: \frac{ydx}{dy} : y^2$, oritur exinde æquatio ad Parabolam Conicam; sumendo enim facta mediorum & ultimorum habebitur $dy^2 + yddy = 0$, quæ quidem æquatio, ut in §. III. ostendimus ad Parabolam pertinet Conicam.

§. V.

Problema. *Si in Curva quadam fuerit Radius Curvaturæ ad productum Normalis & Summæ Subtangentis & Subnormalis in eadem ratione, qua Abscissa ad Subtangentem, æquationem Curvæ invenire.*

Ex hypothesi habetur $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-ddydx} : \frac{y^2 . \overline{dx^2 + dy^2}^{\frac{3}{2}}}{dx^2 dy} :: x : \frac{ydx}{dy}$, unde reductione obtinetur æquatio diffe-

ren-

rentio-differentialis $\frac{dx^2}{x} + yddy = 0$. (A). Ut vero hæc integretur, si N fuerit Numerus cujus Logarithmus Hyperbolicus $= 1$, ponatur $x = N^{2\int zdv}$, adeoque $dx = 2N^{2\int zdv} zdv$, & existente dx constante, $ddx = 0 = 4N^{2\int zdv} z^2dv^2 + 2N^{2\int zdv} dzdv + 2N^{2\int zdv} zddv$; erit itaque $ddv = -2zdv^2 - \frac{dzdv}{z}$: ponendo ulterius $y = N^{\int zdv} v$, habebitur $dy = N^{\int zdv} vzdv + N^{\int zdv} dv$ & $ddy = N^{\int zdv}(vz^2dv^2 + 2zdv^2 + vdzdv + zvddv + ddv)$; si vero substituatur valor ipsius $ddv = -2zdv^2 - \frac{dzdv}{z}$ supra inventus, obtinetur facta reductione $ddy = N^{\int zdv}\left(-vz^2dv^2 - \frac{dzdv}{z}\right)$. Hisce autem valoribus ipsarum x, dx, y & ddy in æquatione (A) substitutis, eruitur $N^{2\int zdv} 4z^2dv^2 + N^{2\int zdv} v\left(-vz^2dv^2 - \frac{dzdv}{z}\right) = 0$, & æquatione per $N^{2\int zdv} vz^2dv$ divisa, prodit $\frac{4dv}{v} - vdv = \frac{dz}{z^3}$ cujus integrale est $4Lv - \frac{v^2}{2} = -\frac{1}{2z^2}$. (B). Sumendo autem $x = N^{2\int zdv}$, habebitur $Lx = 2\int zdv$, & $\frac{dx}{x} = 2zdv$; existente porro $y = N^{\int zdv} v$,

erit

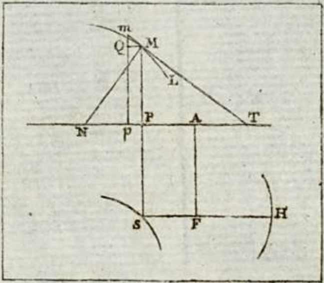
erit $Ly = \int zdv + Lv = \frac{Lx}{2} + Lv$, & transeundo a Logarithmis ad quantitates absolutas $y = x^{\frac{1}{2}}v$, adeoque $v = \frac{y}{x^{\frac{1}{2}}}$, & $dv = \frac{xdy - \frac{1}{2}ydx}{x^{\frac{3}{2}}}$, obtinebitur itaque $z = \frac{dx}{2xdv} = \frac{x^{\frac{1}{2}}dx}{2xdy - ydx}$. Si jam ponantur valores z & v inventi in æquatione (B), habebitur $4l\frac{y}{x^{\frac{1}{2}}} - \frac{y^2}{2x} = -\frac{\overline{2xdy - ydx}^2}{2xdx^2}$, seu $2x.\overline{4Ly - 2Lx} - y^2 = -\frac{\overline{2xdy - ydx}^2}{dx^2}$, adeoque $2xdy - ydx = dx\sqrt{y^2 - 2x.\overline{4Ly - 2Lx}}$, & æquatione per xy divisa $\frac{2dy}{y} - \frac{dx}{x} = \frac{dx}{xy}\sqrt{y^2 - 2x.\overline{4Ly - 2Lx}}$. (C).
Ut vero hujus æquationis pateat integrale, ponatur $4Ly - 2Lx = 2Lp$, unde $2Ly = Lx + Lp$ & $\frac{2dy}{y} = \frac{dx}{x} + \frac{dp}{p}$ atque $y^2 = px$, quibus ipsarum y & dy valoribus in æquatione (C) substitutis, prodit $\frac{dx}{x} = \frac{dp}{\sqrt{p^2 - 4pLp}}$, & hinc $Lx = \int\frac{dp}{\sqrt{p^2 - 4pLp}}$. Alterius vero æquationis membri integrale ope Seriei in-

infinitæ investigari potest, ipsum vero calculum, angustia temporis circumscripti, jam omittimus. Inventa autem methodo jam dicta $\int \frac{dp}{\sqrt{p^2 - 4pLp}}$, invenitur æquatio Curvæ restituto valore ipsius $p = \frac{y^2}{x}$.

CORRIGENDA.

Pag. 5. L. ult. loco $\frac{\frac{1}{2}\overline{a + x}dx}{a + x}$ leg. $\frac{\frac{1}{2}\overline{a + x}.dx}{(a + x)x}$.

Pag. 6. L. 10. loco dy leg. dx.



C L S sc.